

On supermembrane actions on Calabi-Yau 3-folds

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Abstract

In this note we examine the supermembrane action on Calabi-Yau 3-folds. We write down the Dirac-Born-Infeld part of the action, and show that it is invariant under the rigid spacetime supersymmetry.

1 Introduction

The study of p -branes as solitonic solutions of supergravity has now become indispensable for the understanding of nonperturbative effects in superstring theory. On flat spacetimes p -branes are described by an action which consists of two parts; the Dirac-Born-Infeld (DBI) part, and the Wess-Zumino part. The latter can only be written down for some specific spacetime dimensions. For supermembranes, for instance, it exists only for 4, 5, 7, and 11 spacetime dimensions [1]. Both parts of the action are invariant under the rigid spacetime supersymmetry transformations, however, for a specific choice of normalization of the Wess-Zumino term, the whole action turns out to have an extra local symmetry known as κ -symmetry. The κ -symmetry allows one to remove the redundant fermionic degrees of freedom leaving only the physical ones.

p -branes (and also D-branes) have so far been studied on flat spacetimes, and in some cases on AdS spaces (look at [1, 2] and the references therein, for instance). On Calabi-Yau manifolds, on the other hand, the action is not known, though, in some cases they have been studied through their low energy effective action [3, 4], which for Dp -branes is just the dimensional reduction of 10d super Yang-Mills theory down to $p + 1$ dimensions [5]. Along these lines, an effective action of D-branes on Calabi-Yau 3-folds was also introduced in [6].

The study of p -branes on Calabi-Yau 3-folds becomes crucial if we are to learn about the nonperturbative effects of superstring theory or M-theory compactified on such manifolds.

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So it is of importance to find the effective action of p -branes on Calabi-Yau manifolds. This note is an attempt to construct the action of supermembranes on Calabi-Yau manifolds. In section 2, we begin with the preliminaries. The field decompositions on Calabi-Yau 3-folds are used to write down the supersymmetric invariant one-forms Π . However, these forms will not be invariant under supersymmetry transformations as soon as we lift them on Calabi-Yau manifolds. We then show that the supersymmetry transformations on a Calabi-Yau 3-fold can be deformed to look like the BRST transformations. This fact is then used to construct the supersymmetric DBI action of supermembranes.

2 Preliminaries and the construction of the action

Consider a Calabi-Yau 3-fold M which has the holonomy group $SU(3)$. Take θ to be the singlet spinor on M with the negative chirality. Then, as in [7, 6], we can use the following Fierz identities

$$\theta\theta^\dagger + \frac{1}{2}\gamma^{\bar{\alpha}}\theta^*\theta^t\gamma_{\bar{\alpha}} = \frac{1}{2}(1 - \gamma_7) \quad (1)$$

$$\theta^*\theta^t + \frac{1}{2}\gamma^\alpha\theta\theta^\dagger\gamma_\alpha = \frac{1}{2}(1 + \gamma_7) \quad (2)$$

to decompose a complex spinor $\Psi = \Psi_L + \Psi_R$ on M . Here we have used $\alpha, \beta, \gamma, \dots$ to represent the complex tangent indices on M . This results in

$$\begin{aligned} \Psi_L &= \theta\psi + \frac{1}{2}\gamma^{\bar{\alpha}}\theta^*\psi_{\bar{\alpha}} \Rightarrow \Psi_L^\dagger = \bar{\psi}\theta^\dagger + \frac{1}{2}\bar{\psi}_{\bar{\alpha}}\theta^t\gamma^{\bar{\alpha}} \\ \Psi_R &= \theta^*\chi + \frac{1}{2}\gamma^\alpha\theta\chi_\alpha \Rightarrow \Psi_R^\dagger = \bar{\chi}\theta^t + \frac{1}{2}\bar{\chi}_{\bar{\alpha}}\theta^\dagger\gamma^{\bar{\alpha}}, \end{aligned} \quad (3)$$

where we have defined

$$\begin{aligned} \psi &= \theta^\dagger\Psi_L, & \psi_{\bar{\alpha}} &= \theta^t\gamma_{\bar{\alpha}}\Psi_L \\ \chi &= \theta^t\Psi_R, & \chi_\alpha &= \theta^\dagger\gamma_\alpha\Psi_R. \end{aligned}$$

As θ is a singlet, we have chosen $\gamma_\alpha\theta = 0$. The covariantly constant forms that can be constructed from θ are the Kähler 2-form $k_{\alpha\bar{\beta}} = i\theta^\dagger\gamma_{\alpha\bar{\beta}}\theta$, and the holomorphic 3-form $C_{\alpha\beta\gamma} = \theta^\dagger\gamma_{\alpha\beta\gamma}\theta^*$.

As mentioned in the introduction, the supermembrane action can exist in seven dimensions. Therefore we first write the action on flat \mathbf{R}^7 spacetime [1] using the above decomposition of fields, then we lift it on to $M \times \mathbf{R}^1$, where \mathbf{R}^1 represent the time direction.

Let us first recall the construction of the action on flat spacetime [1]. Let $m, n, \dots = 0, 1, 2, \dots, 6$ represent the tangent indices on the whole manifold, and $\mu, \nu, \dots = 1, 2, \dots, 6$ indicate the indices on M . $i, j, k, \dots = 1, 2, 3$ will be the tangent indices on the worldvolume of the brane. As for the gamma matrices we take

$$\gamma_6 = \gamma_0\gamma_1 \cdots \gamma_5, \quad \gamma_7 = i\gamma_0,$$

where γ_μ 's are hermitian and $\gamma_0^\dagger = -\gamma_0$. Define

$$\Pi^m = dX^m - i\bar{\Psi}\gamma^m d\Psi + id\bar{\Psi}\gamma^m\Psi,$$

where $\bar{\Psi} = \Psi^\dagger\gamma_0$. Π^m is invariant under the following rigid supersymmetry transformations

$$\delta X^m = i\bar{\varepsilon}\gamma^m\Psi - i\bar{\Psi}\gamma^m\varepsilon, \quad \delta\Psi = \varepsilon. \quad (4)$$

Here $\varepsilon = \varepsilon_L + \varepsilon_R$ is a constant complex six-dimensional spinor. On M , ε decomposes just like Ψ , however, as we are interested in the scalar part of the transformations, we drop the triplet part of the ε ;

$$\varepsilon_L = \theta\epsilon, \quad \varepsilon_R = \theta^*\eta. \quad (5)$$

Using the field decompositions in (3) and (5), the supersymmetry transformations (4) read (with $\phi \equiv X^0$)

$$\begin{aligned} \delta X^\alpha &= \bar{\eta}\psi^\alpha + \epsilon\bar{\chi}^\alpha \\ \delta X^{\bar{\alpha}} &= -\eta\bar{\psi}^{\bar{\alpha}} - \bar{\epsilon}\chi^{\bar{\alpha}} \\ \delta\phi &= i\bar{\epsilon}\psi + i\epsilon\bar{\psi} + i\bar{\eta}\chi + i\eta\bar{\chi} \\ \delta\psi &= \epsilon, \quad \delta\chi = \eta \\ \delta\psi^\alpha &= 0, \quad \delta\chi^{\bar{\alpha}} = 0. \end{aligned} \quad (6)$$

For the sake of simplicity, in the following, we set $\eta = 0$ in the above transformations and work with just one of the supersymmetries survived on M .

The components of Π , on the other hand, are

$$\begin{aligned} \Pi^\alpha &= dX^\alpha - \bar{\chi}D\psi^\alpha - \psi^\alpha d\bar{\chi} - \psi D\bar{\chi}^\alpha - \bar{\chi}^\alpha d\psi + \frac{1}{4}C^{\alpha\beta\gamma}(\bar{\psi}_\beta D\chi_\gamma - \chi_\beta D\bar{\psi}_\gamma) \\ \Pi^{\bar{\alpha}} &= dX^{\bar{\alpha}} + \chi D\bar{\psi}^{\bar{\alpha}} + \bar{\psi}^{\bar{\alpha}} d\chi + \bar{\psi} D\chi^{\bar{\alpha}} + \chi^{\bar{\alpha}} d\bar{\psi} - \frac{1}{4}C^{\bar{\alpha}\bar{\beta}\bar{\gamma}}(\psi_{\bar{\beta}} D\bar{\chi}_{\bar{\gamma}} - \bar{\chi}_{\bar{\beta}} D\psi_{\bar{\gamma}}) \\ \Pi^0 &= d\phi - i\bar{\psi}d\psi - i\bar{\chi}d\chi - \frac{i}{2}g_{\alpha\bar{\beta}}\bar{\psi}^{\bar{\beta}}D\psi^\alpha - \frac{i}{2}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha D\chi^{\bar{\beta}} \\ &\quad - i\psi d\bar{\psi} - i\chi d\bar{\chi} - \frac{i}{2}g_{\alpha\bar{\beta}}\psi^\beta D\bar{\psi}^{\bar{\alpha}} - \frac{i}{2}g_{\alpha\bar{\beta}}\chi^{\bar{\alpha}} D\bar{\chi}^{\bar{\beta}}, \end{aligned}$$

where $D\psi^\alpha = d\psi^\alpha + dX^\beta\Gamma_{\beta\gamma}^\alpha\psi^\gamma$. Further define

$$M_{ij} = \delta_{mn}\Pi_i^m\Pi_j^n. \quad (7)$$

Now as Π 's are invariant under the SUSY transformations on flat spacetime, the action

$$S_{DBI} = -T \int d^3\sigma \sqrt{\det M_{ij}}, \quad (8)$$

is also trivially invariant.

On a Calabi-Yau 3-fold, however, neither Π nor the metric is invariant under the SUSY transformations. Therefore $M_{ij} = g_{mn}\Pi_i^m\Pi_j^n$ will not be invariant;

$$\delta(g_{mn}\Pi_i^m\Pi_j^n) = \Delta(g_{mn}\Pi_i^m\Pi_j^n) = g_{mn}(\Delta\Pi_i^m)\Pi_j^n + g_{mn}\Pi_i^m\Delta\Pi_j^n, \quad (9)$$

where the covariant variation is defined by

$$\Delta\Pi^\alpha = \delta\Pi^\alpha + \delta X^\rho \Gamma_{\rho\sigma}^\alpha \psi^\sigma,$$

and the last equality in (9) follows as the metric g_{mn} is covariantly constant.

To get a supersymmetric action on $M \times \mathbf{R}^1$, firstly we note that on *flat* spacetimes the transformations (6) square to zero (BRST-like) when acting on any field except ϕ . Secondly if we define the operator \tilde{Q} by

$$\tilde{Q} = Q + \bar{Q} \quad , \quad \delta = i\epsilon Q + i\bar{\epsilon}\bar{Q},$$

we can see that M_{ij} in the action can be written as a BRST-exact term;

$$M_{ij} = \delta_{mn} \Pi_i^m \Pi_j^n = \{\tilde{Q}, \frac{i}{2}(\psi + \bar{\psi})\delta_{mn} \Pi_i^m \Pi_j^n\}. \quad (10)$$

This follows as $\{\tilde{Q}, \frac{i}{2}(\psi + \bar{\psi})\} = 1$. If we could maintain the BRST property of \tilde{Q} on M , with a choice of M_{ij} as in (10), it would be straightforward to construct the supersymmetric action on a Calabi-Yau 3-fold. All we need to do is to compute the term

$$\{\tilde{Q}, \frac{i}{2}(\psi + \bar{\psi})g_{mn} \Pi_i^m \Pi_j^n\} \quad (11)$$

to get M_{ij} on $M \times \mathbf{R}^1$. The invariance of the action under \tilde{Q} then follows as $\tilde{Q}^2 = 0$;

$$\begin{aligned} \{\tilde{Q}, S_{DBI}\} &= -\frac{T}{2} \int d^3\sigma \sqrt{\det M_{ij}} M^{ij} \{\tilde{Q}, M_{ij}\} \\ &= -\frac{T}{2} \int d^3\sigma \sqrt{\det M_{ij}} M^{ij} \{\tilde{Q}, \{\tilde{Q}, \frac{i}{2}(\psi + \bar{\psi})g_{mn} \Pi_i^m \Pi_j^n\}\} = 0. \end{aligned} \quad (12)$$

In the following, we will see how this method works.

First of all, on $M \times \mathbf{R}^1$ we need to covariantize the transformations (6). The only transformation which needs to change is that of ψ^α . So we write

$$\delta\psi^\alpha = -\delta X^\rho \Gamma_{\rho\sigma}^\alpha \psi^\sigma.$$

With this change, however, δ does not square to zero anymore when acting on ψ^α . To maintain this property of δ , we add another term proportional to the Riemann tensor as follows

$$\delta\psi^\alpha = -\epsilon \bar{\chi}^\rho \Gamma_{\rho\sigma}^\alpha \psi^\sigma - i\epsilon \bar{\psi} \bar{\chi}^\beta \chi^{\bar{\rho}} \psi^\gamma R_{\beta\bar{\rho}\gamma}^\alpha,$$

or

$$\Delta\psi^\alpha = -i\epsilon \bar{\psi} \bar{\chi}^\beta \chi^{\bar{\rho}} \psi^\gamma R_{\beta\bar{\rho}\gamma}^\alpha.$$

Noting that on Kähler manifolds $R_{\beta\rho\gamma}^\alpha = 0$ and $R_{\beta\bar{\rho}\gamma}^\alpha = R_{\gamma\bar{\rho}\beta}^\alpha$, it is easy to check that with this change $\delta_{\epsilon_1} \delta_{\epsilon_2} \psi^\alpha = 0$.

Taking M_{ij} as in (11) we need to work out the variation of Π 's. Since the holomorphic 3-form $C^{\alpha\beta\gamma}$ is covariantly constant, we obtain

$$\begin{aligned}\Delta\Pi_i^\alpha &= -\bar{\chi}\Delta(D_i\psi^\alpha) + \partial_i\bar{\chi}\Delta\psi^\alpha - \psi\Delta(D_i\bar{\chi}^\alpha) + \frac{1}{4}C_{\bar{\beta}\bar{\gamma}}^\alpha \left(\Delta\bar{\psi}^{\bar{\beta}}D_i\bar{\chi}^{\bar{\gamma}} + \bar{\psi}^{\bar{\beta}}\Delta(D_i\bar{\chi}^{\bar{\gamma}}) - \bar{\chi}^{\bar{\beta}}\Delta(D_i\bar{\psi}^{\bar{\gamma}}) \right) \\ \Delta\Pi_i^0 &= -\frac{i}{2}g_{\alpha\bar{\beta}}(\Delta\bar{\psi}^{\bar{\beta}}D_i\psi^\alpha + \bar{\psi}^{\bar{\beta}}\Delta(D_i\psi^\alpha) + \bar{\chi}^\alpha\Delta(D_i\bar{\chi}^{\bar{\beta}})) + \text{h.c.},\end{aligned}$$

where

$$\begin{aligned}\Delta(D_i\psi^\alpha) &= -(\epsilon\bar{\chi}^\beta\psi^\sigma X_i^{\bar{\rho}} + \bar{\epsilon}\chi^{\bar{\rho}}\psi^\sigma X_i^\beta)R_{\beta\bar{\rho}\sigma}^\alpha - i\epsilon D_i(\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\gamma R_{\beta\bar{\rho}\gamma}^\alpha) \\ \Delta(D_i\bar{\chi}^\alpha) &= -\bar{\epsilon}X_i^\beta\chi^{\bar{\rho}}\bar{\chi}^\gamma R_{\beta\bar{\rho}\gamma}^\alpha.\end{aligned}\tag{13}$$

So finally we find the following supersymmetric action for membranes on $M \times \mathbf{R}^1$

$$S_{DBI} = -T \int d^3\sigma \sqrt{\det M_{ij}},\tag{14}$$

with

$$\begin{aligned}M_{ij} &= g_{mn}\Pi_i^m\Pi_j^n \\ &- \frac{i}{2}(\psi + \bar{\psi})g_{\alpha\bar{\beta}}\Pi_{(j}^{\bar{\beta}} \left[(i\bar{\chi}\bar{\chi}^\beta\psi^\sigma X_i^{\bar{\rho}} + i\bar{\chi}\chi^{\bar{\rho}}\psi^\sigma X_i^\beta) \right. \\ &+ \partial_i\bar{\chi}\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\sigma + i\psi\chi^{\bar{\rho}}\bar{\chi}^\sigma X_i^\beta)R_{\beta\bar{\rho}\sigma}^\alpha - \bar{\chi}D_i(\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\gamma R_{\beta\bar{\rho}\gamma}^\alpha) \\ &+ \frac{1}{4}C_{\bar{\beta}\bar{\gamma}}^\alpha \left((\psi\chi^{\bar{\sigma}}\bar{\chi}^{\bar{\rho}}\bar{\psi}^{\bar{\eta}}D_i\bar{\chi}^{\bar{\gamma}} + i\bar{\psi}^{\bar{\gamma}}\bar{\chi}^{\bar{\rho}}\chi^{\bar{\eta}}X_i^{\bar{\sigma}} \right. \\ &- i\chi^{\bar{\gamma}}\chi^{\bar{\sigma}}\bar{\psi}^{\bar{\eta}}X_i^{\bar{\rho}} - i\chi^{\bar{\gamma}}\bar{\chi}^{\bar{\rho}}\bar{\psi}^{\bar{\eta}}X_i^{\bar{\sigma}})R_{\bar{\sigma}\bar{\rho}\bar{\eta}}^{\bar{\beta}} + \chi^{\bar{\beta}}D_i(\psi\chi^{\bar{\sigma}}\bar{\chi}^{\bar{\rho}}\bar{\psi}^{\bar{\eta}}R_{\bar{\sigma}\bar{\rho}\bar{\eta}}^{\bar{\gamma}}) \left. \right) \left. \right] \\ &+ \frac{i}{4}(\psi + \bar{\psi})g_{\alpha\bar{\beta}}\Pi_{(j}^0 \left[-i\psi\chi^{\bar{\sigma}}\bar{\chi}^{\bar{\rho}}\bar{\psi}^{\bar{\gamma}}D_i\psi^\alpha R_{\bar{\sigma}\bar{\rho}\bar{\gamma}}^{\bar{\beta}} \right. \\ &- \bar{\psi}^{\bar{\beta}}(\bar{\chi}^\sigma\psi^\gamma X_i^{\bar{\rho}} + \chi^{\bar{\rho}}\psi^\gamma X_i^\sigma)R_{\sigma\bar{\rho}\gamma}^\alpha - i\bar{\psi}^{\bar{\beta}}D_i(\bar{\psi}\bar{\chi}^\beta\chi^{\bar{\rho}}\psi^\gamma R_{\beta\bar{\rho}\gamma}^\alpha) \left. \right] + \text{h.c.}.\end{aligned}$$

As mentioned earlier, since $\delta_{\epsilon_2}\delta_{\epsilon_1}$ acting on any field¹ gives zero it follows that $\tilde{Q}^2 = 0$, which ensures the invariance of the action under \tilde{Q} .

Apart from the DBI action which was constructed above, supermembranes action on flat spacetime has another part, the Wess-Zumino term. One way to obtain the WZ term is to vary the DBI action with respect to the κ -symmetry transformations. One then looks for a supersymmetric (up to a total derivative) 3-form on the worldvolume such that its variation under the κ -transformations cancels the variation of DBI action. The whole action $S = S_{DBI} + S_{WZ}$ is then invariant under both the supersymmetry and κ -symmetry transformations. We hope to return to these issues in future works.

¹ δ^2 acting on ϕ does not give zero, but this does not cause any harm as this field appears in Π^0 as $d\phi$ and δ^2 acting on Π^0 still gives zero.

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